

A RIGIDITY RESULT FOR DIMENSION DATA

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ABSTRACT. The dimension datum of a closed subgroup of a compact Lie group is a sequence by assigning the invariant dimension of each irreducible representation restricting to the subgroup. We prove that any sequence of dimension data contains a converging sequence with limit the dimension datum of a subgroup interrelated to subgroups giving this sequence. This rigidity has an immediate corollary that the space of dimension data of closed subgroups in a given compact Lie group is sequentially compact. Moreover, we deduce some known finiteness and rigidity results from it, as well as give an application to isospectral geometry.

1. INTRODUCTION

Given a compact Lie group G , let \widehat{G} be the set of equivalence classes of irreducible finite-dimensional complex linear representations of G . We endow the space $\mathbb{Z}^{\widehat{G}}$ with the product topology of the discrete topologies on the factors \mathbb{Z} . Given a closed subgroup H of G , the function

$$\mathcal{D}_H : \widehat{G} \rightarrow \mathbb{Z}, V \mapsto \dim V^H$$

is an element of $\mathbb{Z}^{\widehat{G}}$, where V^H is the linear space of H -invariant vectors in a complex linear representation V of G . We call \mathcal{D}_H the *dimension datum* of H . In this way we have a map

$$\mathcal{D} : \{\text{closed subgroup of } G\} / (G - \text{conjugacy}) \rightarrow \mathbb{Z}^{\widehat{G}}, H \mapsto \mathcal{D}_H.$$

Dimension data arise from number theory in the determination of monodromy groups (cf. [K]). They also appear in differential geometry: in relation with the spectra of the Laplace operators on Riemannian homogeneous spaces (cf. [Sut]). Moreover, Langlands has suggested to use the dimension data as a key ingredient in his program “Beyond Endoscopy” (cf. [La], Page 2, Problem (II)). In the literature, dimension data have been studied in the articles [LP], [Lar] and [AYY].

Endowing the space $\mathbb{Z}^{\widehat{G}}$ (identified with the set of all maps from \widehat{G} to \mathbb{Z}) with the product topology of the discrete topologies on the factors \mathbb{Z} , we could consider the convergence of sequences of dimension data. Since $0 \leq \dim V^H \leq \dim V$ for any complex irreducible representation V of G and any closed subgroup H of G , each sequence of dimension data $\{\mathcal{D}_{H_i} : i \geq 1\}$ contains a convergent subsequence as elements in $\mathbb{Z}^{\widehat{G}}$. This raises a question: is the limit still a dimension datum \mathcal{D}_H and does $\{H_i : i \geq 1\}$ give some restriction on H ? Theorem 1.1 answers this question completely. It states that the limit must be a dimension datum \mathcal{D}_H and the subgroup H is strongly controlled by a subsequence $\{H_{i_j} : j \geq 1\}$. We also give some consequences of Theorem 1.1 by strengthening a theorem of Larsen and give different proofs of two theorems proving in a joint paper [AYY], as well as giving an application to isospectral geometry.

Theorem 1.1. *Given a compact Lie group G , let $\{H_n \mid n \geq 1\}$ be a sequence of closed subgroups of G . Then there exist two closed subgroups H, H' of G with $[H'_0, H'_0] \subset H \subset H'$*

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and H' quasi-primitive, a subsequence $\{H_{n_j} \mid j \geq 1\}$ and a sequence $\{g_j \mid j \geq 1, g_j \in G\}$, such that

$$\forall j \geq 1, [H'_0, H_0] \subset g_j H_{n_j} g_j^{-1} \subset H$$

and

$$\lim_{j \rightarrow \infty} \mathcal{D}_{H_{n_j}} = \mathcal{D}_H.$$

This theorem has an immediate corollary.

Corollary 1.2. *The subset $\text{Im}(\mathcal{D}) \subset \mathbb{Z}^{\widehat{G}}$ is a closed subset of $\mathbb{Z}^{\widehat{G}}$.*

Corollary 1.2 can be equivalently stated as follows.

Corollary 1.3. *The topological space $\text{Im}(\mathcal{D})$ is sequentially compact.*

Here the topology on $\text{Im}(\mathcal{D})$ means the subspace topology inherited from the topology on $\mathbb{Z}^{\widehat{G}}$.

Theorem 1.1 is equivalent to the combination of the following two theorems. Theorem 1.4 is a statement about dimension data, and Theorem 1.5 is an extension of Theorem 2.2 to non-finite closed subgroups.

Theorem 1.4. *Given a compact Lie group G , let $\{H_n \mid n \geq 1\}$ be a sequence of closed subgroups of G . Then there exists a closed subgroup H of G , a subsequence $\{H_{n_j} \mid j \geq 1\}$ and a sequence $\{g_j \mid j \geq 1, g_j \in G\}$, such that*

$$\forall j \geq 1, [H_0, H_0] \subset g_j H_{n_j} g_j^{-1} \subset H$$

and

$$\lim_{j \rightarrow \infty} \mathcal{D}_{H_{n_j}} = \mathcal{D}_H.$$

Theorem 1.5. *Given a compact Lie group G and a closed subgroup H , there exist a quasi-primitive subgroup H' of G such that $[H'_0, H'_0] \subset H \subset H'$.*

Notation and conventions. Given a compact Lie group G ,

- (1) let G_0 be the connected component of G containing the identity element and $[G, G]$ the commutator subgroup.
- (2) Write G^\natural for the set of conjugacy classes in G .
- (3) Denote by μ_G the unique Haar measure on G with $\int_G 1 \mu_G = 1$.
- (4) Write \widehat{G} for the set of equivalence classes of irreducible finite-dimensional complex linear representations of G .
- (5) Denote by V^G the subspace of G -invariant vectors in a complex linear representation V of G .

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2. PRIMITIVE AND QUASI-PRIMITIVE SUBGROUPS

In [Yu], we define primitive and quasi-primitive subgroups and prove a generalization of a theorem of Borel and Serre by using them. In this section, we recall the definition and some results in that paper. Lemma 2.3, Proposition 2.4 and Lemma 2.5 are used in the proof of the main theorem of this paper. Lemma 2.5 is due to A. Weil (cf. []).

Definition 2.1. A closed subgroup H of a compact Lie group G is called *Lie primitive* if $(A_G)_0 \subset H$ and for any closed subgroup K of G containing H with K_0 normalizing H_0 , either $[G : K]$ is finite or $[K : H]$ is finite. It is called *Lie quasi-primitive* if there exists a sequence of closed subgroups of G , $G = H_0 \supset H_1 \supset \dots \supset H_s = H$, such that each H_{i+1} is Lie primitive in H_i , $0 \leq i \leq s-1$.

Theorem 2.2. Given a finite subgroup S of a compact Lie group G , there exists a Lie quasi-primitive subgroup H of G containing S and satisfying that:

- (1) H_0 is a Cartan subgroup of G , or
- (2) H_0 is not abelian and SA_H/A_H is a Lie primitive subgroup of H/A_H .

Lemma 2.3. Given a compact semisimple Lie group G and a non-primitive closed subgroup S , there exists a primitive subgroup H of G containing S and such that $\dim H < \dim G$, and S_0 is a proper normal subgroup of H_0 .

Proposition 2.4. Given a compact Lie group G , there exist only finitely many conjugacy classes of Lie primitive and Lie quasi-primitive subgroups of G .

Lemma 2.5. Given a finite group S and a compact semisimple Lie group G , the number of conjugacy classes of homomorphisms from S to G is finite.

3. PROOF OF THE MAIN THEOREM

An element in $\mathbb{Z}^{\widehat{G}}$ can be expressed as a sequence $\vec{n} = \{a_\rho \mid \rho \in \widehat{G}\}$, or equivalently as a function $\vec{n} : \widehat{G} \rightarrow \mathbb{Z}$,

$$\vec{n}(\rho) = a_\rho, \forall \rho \in \widehat{G}.$$

In the topological space $\mathbb{Z}^{\widehat{G}}$, a sequence $\{\vec{n}_i \mid i \geq 1\}$ converges to \vec{n} if and only if for any finite subset $\{\rho_j \mid 1 \leq j \leq m\}$ of \widehat{G} , we have

$$\lim_{i \rightarrow \infty} \vec{n}_i(\rho_j) = \vec{n}(\rho_j)$$

for each j , $1 \leq j \leq m$.

Let $X_G = \{\vec{n} \in \mathbb{Z}^{\widehat{G}} \mid 0 \leq \vec{n}(\rho) \leq \dim \rho, \forall \rho \in \widehat{G}\}$. Then $\mathcal{D}_H \in X_G$ for each closed subgroup H of G . Since X_G is a closed compact subset, we have the following proposition.

Proposition 3.1. Any sequence $\{H_n \mid n \geq 1\}$ of closed subgroups of G contains a subsequence $\{H_{n_j} \mid j \geq 1\}$ such that $\mathcal{D}_{H_{n_j}}$ converges to some element n in X_G .

Proof of Theorem 1.1. We prove it by induction on $\dim G$.

Step 1. Reduction to the case that $[G_0, G_0] \subset H_n$ for any n .

Let \mathfrak{g}_0 be the Lie algebra of G and $\mathfrak{u}_0 = [\mathfrak{g}_0, \mathfrak{g}_0]$ the derived subalgebra of \mathfrak{g}_0 . Then \mathfrak{u}_0 is a compact semisimple Lie algebra. There is an adjoint homomorphism

$$\pi : G \rightarrow \text{Aut}(\mathfrak{u}_0).$$

Write $G' = \text{Aut}(\mathfrak{u}_0)$. It is a compact semisimple Lie group of adjoint type.

Assume first there are infinitely many n such that $\dim \pi(H_n) \neq \dim G'$, by replacing $\{H_n \mid i \geq 1\}$ by a subsequence if necessary, we may assume that $\dim \pi(H_n) < \dim G'$ for any $n \geq 1$. By Lemma 2.3, each $\pi(H_n)$ is contained in a primitive subgroup of G' with a lower dimension. As primitive subgroups of G' have only finitely many conjugacy classes (cf. Proposition 2.4), we may assume that all $\pi(H_n)$ are contained in a primitive subgroup G_1 of G' with $\dim G_1 < \dim G'$. Write $G_2 = \pi^{-1}(G_1)$. Since $\pi(H_n) \subset G_1$, we have $H_n \subset G_2$ for any n . We finish the proof in this case by induction.

Now assume there are only finitely many n such that $\dim \pi(H_n) \neq \dim G'$, by replacing $\{H_n \mid n \geq 1\}$ by a subsequence if necessary, we may assume that $\dim \pi(H_n) = \dim G'$ for any $n \geq 1$. For any n , from $\dim \pi(H_n) = \dim G'$ we get

$$\dim \text{Int}(\mathfrak{u}_0) = \dim G' = \dim \pi(H_n) = \dim \pi(H_n \cap G_0).$$

Thus $\pi(H_n \cap G_0) = \text{Int}(\mathfrak{u}_0)$ since $\pi(H_n \cap G_0) \subset \text{Int}(\mathfrak{u}_0)$ and $\text{Int}(\mathfrak{u}_0)$ is connected.

We have $G_0 = Z_{G_0}[G_0, G_0]$, $G_0/Z_{G_0} \cong \text{Int}(\mathfrak{u}_0)$ and $\pi : G_0 \rightarrow \text{Int}(\mathfrak{u}_0)$ is just the projection map $G_0 \rightarrow G_0/Z_{G_0}$. For any n , since $\pi(H_n \cap G_0) = \text{Int}(\mathfrak{u}_0)$, we have $(H_n \cap G_0)Z_{G_0} = G_0$. Thus

$$[G_0, G_0] = [(H_n \cap Z_{G_0})Z_{G_0}, H_n \cap Z_{G_0})Z_{G_0}] = [H_n \cap Z_{G_0}, H_n \cap Z_{G_0}] \subset H_n.$$

From now on, we assume that $[G_0, G_0] \subset H_n$ for any n .

Step 2. Reduction to the case that G_0 is abelian.

Given a $\rho \in \widehat{G}$, if $\rho|_{[G_0, G_0]}$ does not contain any non-zero trivial subrepresentation, we have

$$\mathcal{D}_{H_n}(\rho) = \dim V_\rho^{H_n} = 0.$$

If $\rho|_{[G_0, G_0]}$ contains a non-zero trivial subrepresentation, ρ is a subrepresentation of

$$\text{Ind}_{[G_0, G_0]}^G(\mathbf{1}_{[G_0, G_0]}).$$

Here $\mathbf{1}_{[G_0, G_0]}$ is the 1-dimensional trivial representation of $[G_0, G_0]$. The action of $[G_0, G_0]$ on ρ is trivial because $[G_0, G_0]$ is a normal subgroup of G . Hence ρ factors through the homomorphism $G \rightarrow G/[G_0, G_0]$. We only need to consider dimension data of the subgroups $\{H_n/[G_0, G_0] \mid n \geq 1\}$ of $G/[G_0, G_0]$. As $(G/[G_0, G_0])_0$ is abelian, we reduce it to the case that G_0 is abelian.

From now on, we assume that G_0 is abelian.

Step 3. Reduction to the case that $\{H_n \cap G_0 \mid n \geq 1\}$ are all equal to some $Z' \subset G_0$ and Z' is a normal subgroup of G .

Set $N_n = H_n/(H_n \cap C_G(G_0))$. Then $N_n \subset G/C_G(G_0)$. Since $G/C_G(G_0)$ is a finite group, there are only finitely many possibilities for N_n . We may assume that they are all equal to some $N \subset G/C_G(G_0)$. The finite group N acts on G_0 by conjugation.

Consider the sequence $\{H_n \cap G_0 \mid n \geq 1\}$. The sequence of dimension data associated to it contains a convergent subsequence by Proposition 3.1. We may assume that the sequence of dimension data itself converges. Denote by L be the dual group of G_0 , i.e. the set of isomorphism classes of 1-dimensional complex linear representations of G_0 . Then L is a lattice. Write $L' \subset L$ for the subset of irreducible representations ρ of G_0 such that

$$\lim_{n \rightarrow \infty} \dim \rho^{H_n \cap G_0} = 1.$$

Claim 3.2. *L' is an N -stable sublattice.*

We defer the proof of Claim 3.2 and finish the proof of Theorem 1.1 first. Let $Z' \subset G_0$ be the intersection of the kernels of the linear characters in L' . By Claim 3.2, L' is N -stable. Thus Z' is also N -stable. Choose a basis $\{\rho_j \mid 1 \leq j \leq m\}$ of L' . For any j , there exists an k_j such that $\dim \rho_j^{H_n \cap G_0} = 1$ for any $n \geq k_j$. Write

$$k = \max\{k_j : 1 \leq j \leq m\}.$$

For any $n \geq k$ and any j with $1 \leq j \leq m$, we have $\dim \rho_j^{H_n \cap G_0} = 1$. That is to say, $H_n \cap G_0$ acts trivially on ρ_j . As ρ_j is a basis of L' , we have $H_n \cap G_0$ acts trivially on any $\rho \in L'$. This means

$$H_n \cap G_0 \subset Z'$$

for any $n \geq k$. Replacing $\{H_n \cap G_0 \subset Z' \mid n \geq 1\}$ by a subsequence if necessary, we may assume that $H_n \cap G_0 \subset Z'$ for any $n \geq 1$. Since Z' is N -stable and $N_n = N$, we have $H_n \subset N_G(Z')$. Replacing G by $N_G(Z')$, we may and do assume that Z' is normal in G . Let $H'_n = H_n Z'$.

Given a $\sigma \in \widehat{G}$, if $\sigma|_{G_0}$ does not contain any simple irreducible sub-representation isomorphic to an element of L' , then

$$\lim_{n \rightarrow \infty} \dim \rho^{H'_n} = \lim_{n \rightarrow \infty} \dim \rho^{H_n} = 0.$$

If $\sigma|_{G_0}$ contains a simple irreducible subrepresentation isomorphic to an element of L' , the action of Z' on σ must be trivial. Thus

$$\lim_{n \rightarrow \infty} \dim \rho^{H'_n} = \lim_{n \rightarrow \infty} \dim \rho^{H_n Z'} = \lim_{n \rightarrow \infty} \dim \rho^{H_n}.$$

By the above,

$$\lim_{n \rightarrow \infty} \mathcal{D}_{H'_n} = \lim_{n \rightarrow \infty} \mathcal{D}_{H_n}.$$

Replacing $\{H_n \mid n \geq 1\}$ by $\{H'_n \mid n \geq 1\}$, it reduces to the case $\{H_n \cap G_0 \mid n \geq 1\}$ are all equal to some $Z' \subset G_0$ and Z' is a normal subgroup of G .

From now on, we assume that G_0 is abelian, there is a subgroup Z' of G_0 that is a normal subgroup of G , and $H_n \cap G_0 = Z'$ for any $n \geq 1$.

Step 4. Conclusion.

Arguing similarly as in Step 2, for any $\rho \in \widehat{G}$, if $\rho|_{Z'}$ does not contain any non-zero trivial subrepresentation, then $\mathcal{D}_{H_n}(\rho) = \dim V_\rho^{H_n} = 0$ for any $n \geq 1$. If $\rho|_{Z'}$ contains a non-zero trivial subrepresentation, the action of Z' on ρ is trivial. Hence ρ factors through the homomorphism $G \rightarrow G/Z'$. We only need to consider the dimension data of the subgroups $\{H_n/Z'\}$ of G/Z' .

Now each H_n/Z' is a finite group with order bounded by $|G/G_0|$. By Proposition 2.5, there is a subsequence $\{H_{n_j} \mid j \geq 1\}$ of $\{H_n \mid n \geq 1\}$ such that the subgroups in $\{H_{n_j}\}$ are conjugate to each other. Let H be one of $\{H_{n_j}\}$ and $H' = G$. Then they satisfy the desired conclusion of the theorem. Going through the reduction steps, we see that the subgroup H' so constructed is a quasi-primitive subgroup of the original compact Lie group G . \square

Proof of Claim 3.2. Given a $\rho \in L'$, there exists a $k_\rho \geq 1$ such that $\dim V_\rho^{H_n \cap G_0} = 1$ for any $n \geq k_\rho$. As G_0 is abelian, we have $\dim \rho = 1$. The above equality means that $H_n \cap G_0$ acts trivially on ρ for any $n \geq k_\rho$. For another $\rho' \in L'$, there also exists a $k_{\rho'} \geq 1$ such that $H_n \cap G_0$ acts trivially on ρ' for any $n \geq k_{\rho'}$. Set $k = \max\{k_\rho, k_{\rho'}\}$. Then $H_n \cap G_0$ acts trivially on $\rho^{\pm a} \otimes \rho'^{\pm 1}$ for any $n \geq k$. Here ρ^{-1} is the contragredient representation of ρ . Hence we have $\rho^{\pm 1} \otimes \rho'^{\pm 1} \in L'$. Therefore, L' is a sublattice of L .

For any $n \geq 1$, since $N_n = H/(H \cap C_{G_0}) = N$, we get that H_n is stable under the conjugation action of N . Hence the set of those $\rho \in L$ such that $\dim V_\rho^{H_n \cap G_0} = 1$ is also stable under N . Therefore, L' is stable under N . \square

Theorem 1.1 implies Corollary 1.2. By Theorem 1.1, there exists a closed subgroup H of G and a subsequence $\{H_{n_j} \mid j \geq 1\}$ such that

$$\lim_{j \rightarrow \infty} \mathcal{D}_{H_{n_j}} = \mathcal{D}_H.$$

Hence

$$\vec{n} = \lim_{n \rightarrow \infty} \mathcal{D}_{H_n} = \lim_{j \rightarrow \infty} \mathcal{D}_{H_{n_j}} = \mathcal{D}_H.$$

\square

4. SOME CONSEQUENCES

Now we fix a compact Lie group G .

In [Lar], given a finite-dimensional complex linear representation V of G , Larsen studied the moments

$$\dim(V^{\otimes a} \otimes (V^*)^{\otimes b})^G$$

and got some rigidity properties for them. Without loss of generality we may assume that V is a unitary representation of G , it is clear that the moments encode a part of the information of the dimension datum of the subgroup G of $U(V)$. Theorem 4.2 is an analogous statement of Theorem 4.2 in [Lar]. It is clear that Theorem 4.2 implies Theorem 4.2 in [Lar]. We also use Theorem 1.1 to prove some theorems in a joint paper [Ayy]. The proofs here are different and independent with those in [Lar] and [Ayy]. Moreover, we give an application of Theorem 1.4 to isospectral geometry.

Definition 4.1. A compact Lie group H is called a quasi-torus if H_0 is a torus.

In the case that $\{H_n \mid n \geq 1\}$ in Theorem 1.2 are all quasi-tori, we can say more about the closed subgroup H . Theorem 4.2 below strengthens [Lar], Theorem 4.2 by making the convergence valid for dimension data rather than only for moments.

Theorem 4.2. If $\vec{n} \in \mathbb{Z}^{\hat{G}}$ is the limit of a sequence of dimension data of quasi-tori of G , then $\vec{n} = \mathcal{D}_H$ for some quasi-torus H contained in G .

Proof. Denote by

$$\vec{n} = \lim_{n \rightarrow \infty} \mathcal{D}_{H_n}.$$

By Theorem 1.1, there exist two closed subgroups $H, H' \subset G$ with $[H'_0, H'_0] \subset H \subset H'$, a subsequence $\{H_{n_j} \mid j \geq 1\}$ and a sequence $\{g_j \mid j \geq 1, g_j \in G\}$ such that

$$\forall j \geq 1, [H'_0, H'_0] \subset g_j H_{n_j} g_j^{-1} \subset H$$

and

$$\lim_{j \rightarrow \infty} \mathcal{D}_{H_{n_j}} = \mathcal{D}_H.$$

As each H_{n_j} is a quasi-torus and $[H'_0, H'_0] \subset g_j H_{n_j} g_j^{-1}$, we have $[H'_0, H'_0] = 0$. In other words, H'_0 is a torus. Therefore H_0 is a torus since $H \subset H'$. We also have

$$\vec{n} = \lim_{n \rightarrow \infty} \mathcal{D}_{H_n} = \lim_{j \rightarrow \infty} \mathcal{D}_{H_{n_j}} = \mathcal{D}_H.$$

□

Lemma 4.3. Given two closed subgroups H and H' of G , if $H \subset H'$ and $\mathcal{D}_H = \mathcal{D}_{H'}$, then $H = H'$.

Proof. Suppose $H \subset H'$ is a proper inclusion. All H' -invariant functions in the space $\text{Ind}_H^{H'} \mathbf{1}_H = L^2(H'/H)$ are constant functions, so there exists a non-trivial irreducible representation σ of H' appearing in $\text{Ind}_H^{H'} \mathbf{1}_H$. Choosing any irreducible representation ρ of G appearing in $\text{Ind}_H^G \sigma$, then $\sigma \subset \rho|_{H'}$. Thus

$$\dim \rho^H - \dim \rho^{H'} \geq \dim \sigma^H - \dim \sigma^{H'} \geq 1 - 0 > 0.$$

Hence $\mathcal{D}_H \neq \mathcal{D}_{H'}$.

□

Theorem 4.4. Given an element $\vec{n} \in \mathbb{Z}^{\hat{G}}$, there are only finitely many conjugacy classes of closed subgroups H of G satisfying $\mathcal{D}_H = \vec{n}$.

Proof. We prove it by contradiction. Suppose there are infinitely many conjugacy classes of closed subgroups H such that $\mathcal{D}_H = \vec{n}$. Then there exists a sequence of pairwise non-conjugate closed subgroups $\{H_n \mid n \geq 1\}$ of G such that $\mathcal{D}_{H_n} = \vec{n}$ for each $n \geq 1$.

By Theorem 1.1, there exists a closed subgroup H of G , a subsequence $\{H_{n_j} \mid j \geq 1\}$ and a sequence $\{g_j \mid j \geq 1, g_j \in G\}$ such that

$$\forall j \geq 1, g_j H_{n_j} g_j^{-1} \subset H$$

and

$$\lim_{j \rightarrow \infty} \mathcal{D}_{H_{n_j}} = \mathcal{D}_H.$$

As subgroups in the sequence $\{H_{n_j} \mid j \geq 1\}$ are non-conjugate to each other, we have a proper inclusion $g_{j_0} H_{n_{j_0}} g_{j_0}^{-1} \subset H$ for $j_0 = 1$ or 2 . Moreover, we have

$$\mathcal{D}_H = \lim_{j \rightarrow \infty} \mathcal{D}_{H_{n_j}} = \vec{n} = \mathcal{D}_{g_{j_0} H_{n_{j_0}} g_{j_0}^{-1}}.$$

Write $H'' = g_{j_0} H_{n_{j_0}} g_{j_0}^{-1}$. Then $H'' \subset H$ is a proper inclusion and $\mathcal{D}_{H''} = \mathcal{D}_H$. This is in contradiction with Lemma 4.3. \square

Theorem 4.5. *Given a closed subgroup K of G , put $\vec{n} = \mathcal{D}_K$ and $\{K_1, \dots, K_r\} = \mathcal{D}^{-1}(\vec{n})$. Then \vec{n} is an isolated point in $\text{Im}(\mathcal{D})$ if and only if K_1, \dots, K_r are all semisimple groups.*

Proof. We omit the proof for the “only if” part of the statement. There is a nice and short proof of this in [AYY].

For the “if” part, suppose $n = \mathcal{D}_K$ is not an isolated point in $\text{Im}(\mathcal{D})$. Then there exists a sequence $\{H_n \mid n \geq 1\}$ of closed subgroups of G such that $\mathcal{D}_{H_n} \neq \vec{n}$ for any $n \geq 1$ and

$$\lim_{n \rightarrow \infty} \mathcal{D}_{H_n} = \vec{n}.$$

By Theorem 1.1, there exist two closed subgroups $H, H' \subset G$ with $[H'_0, H'_0] \subset H \subset H'$, a subsequence $\{H_{n_j} \mid j \geq 1\}$ and a sequence $\{g_j \mid j \geq 1, g_j \in G\}$ such that

$$\forall j \geq 1, [H'_0, H'_0] \subset g_j H_{n_j} g_j^{-1} \subset H$$

and

$$\lim_{j \rightarrow \infty} \mathcal{D}_{H_{n_j}} = \mathcal{D}_H.$$

From these we get $[H'_0, H'_0] \subset H_0 \subset H'_0$ and $\vec{n} = \mathcal{D}_H$. Thus $H \in \mathcal{D}^{-1}(\vec{n})$. As $\mathcal{D}^{-1}(\vec{n})$ consist of semisimple subgroups, we get that H is semisimple. Together with $[H'_0, H'_0] \subset H_0 \subset H'_0$, we get $[H'_0, H'_0] = H_0$. Hence $H_0 \subset g_j H_{n_j} g_j^{-1} \subset H$ for any $j \geq 1$. Therefore there exists an infinite set n_{j_ν} such that $g_{j_\nu} H_{n_{j_\nu}} g_{j_\nu}^{-1}$ are equal to each other. We may and do assume that $\{H_{n_j} \mid j \geq 1\}$ are equal to each other. Write H'' for one of them. Then $\mathcal{D}_{H''} = \mathcal{D}_H$. By Lemma 4.3 we get $H'' = H$. This is in contradiction with the fact that $\mathcal{D}_{H_n} \neq \vec{n}$ for any $n \geq 1$. \square

Given a closed Riemannian manifold (X, m) , one can define the Laplacian operator on the linear space of L^2 -functions on X . It is a self-adjoint positive definite linear operator, so its spectrum consists of discrete real numbers, which is called the Laplacian Spectrum of (X, m) . Given a compact connected Lie group G with a biinvariant Riemannian metric m , for each closed subgroup H of G the homogeneous space G/H inherits a Riemannian metric, still denoted m . We are interested on the Laplacian spectra of these homogeneous spaces here.

Proposition 4.6. *Given a compact connected Lie group G with a biinvariant Riemannian metric m , there exist finitely many conjugacy classes of closed subgroups H of G with the Laplacian spectra of $(G/H, m)$ equal to a given one.*

Proof. We prove it by contradiction. Given a Laplacian spectrum, suppose there are infinitely many conjugacy classes of closed subgroups H such that the spectra of $(G/H, m)$ equal to a given one. Then there exists a sequence of pairwise non-conjugate closed subgroups $\{H_n \mid n \geq 1\}$ of G such that the Laplacian spectra of $(G/H, m)$ being equal to each other. By Theorem 1.1, there exists a closed subgroup H of G , a subsequence $\{H_{n_j} \mid j \geq 1\}$ and a sequence $\{g_j \mid j \geq 1, g_j \in G\}$ such that

$$\forall j \geq 1, g_j H_{n_j} g_j^{-1} \subset H$$

and

$$\lim_{j \rightarrow \infty} \mathcal{D}_{H_{n_j}} = \mathcal{D}_H.$$

As subgroups in the sequence $\{H_{n_j} \mid j \geq 1\}$ are non-conjugate to each other, we have a proper inclusion $g_{j_0} H_{n_{j_0}} g_{j_0}^{-1} \subset H$ for $j_0 = 1$ or 2 . Write $H'' = g_{j_0} H_{n_{j_0}} g_{j_0}^{-1}$. Since the Laplacian spectrum of $(G/H, m)$ can be calculated from the dimension datum of H , the Laplacian spectrum of $(G/H, m)$ is equal to that of H'' . Hence, $H'' \subset H$ is a proper inclusion and $(G/H, m)$ and $(G/H'', m)$ have the same Laplacian spectrum. This is in contradiction with Lemma 4.3. \square

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